Möbius geometry of surfaces of constant mean curvature 1 in hyperbolic space

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Summary. Various transformations of isothermic surfaces are discussed and their interrelations are analyzed. Applications to cmc-1 surfaces in hyperbolic space and their minimal cousins in Euclidean space are presented: the Umehara-Yamada perturbation, the classical and Bryant's Weierstrass type representations, and the duality for cmc-1 surfaces are interpreted in terms of transformations of isothermic surfaces. A new Weierstrass type representation is introduced and a Möbius geometric characterization of cmc-1 surfaces in hyperbolic space and minimal surfaces in Euclidean space is given.

1. Introduction

This paper aims to establish a relation between two fields of current interest.

On one side, isothermic surfaces and their various transformations gained new interest when an integrable system approach was developed [11]: "Darboux pairs" of isothermic surfaces correspond to "curved flats" in a suitable Grassmannian [6] (cf.[15]), a particularly simple type of integrable system. The "spectral parameter" occuring in the loop group description for curved flats [13] gives rise to a "spectral transformation" for isothermic surfaces. Classically, this transformation was independently discovered by P. Calapso [7], [8] who considered it the most fundamental transformation, and by L. Bianchi [2] who introduced it as the "T-transformation". Later, isothermic surfaces were identified as the only surfaces in conformal geometry allowing a second order deformation [9] —

that turns out to be exactly the T-transformation [19]. Another central role is played by the "Darboux transformation" for isothermic surfaces, first introduced by G. Darboux [12]: a corresponding permutability theorem and a quaternionic calculus [15] are the keys to the theory of "discrete isothermic nets" [4], and their transformations [14], [17].

On the other side, surfaces of constant mean curvature 1 in hyperbolic space became a field of interest, complementing research on minimal surfaces in Euclidean space, after R. Bryant [5] developed a Weierstrass type representation for them. Subsequently, M. Umehara and K. Yamada described these surfaces (and the corresponding Weierstrass representation) as a deformation of minimal surfaces in Euclidean space (and their Weierstrass representation) [23], and introduced a notion of duality [24] for cmc-1 surfaces in H^3.

In section 3 of the present paper, we describe the various transformations of isothermic surfaces in terms of a quaternionic calculus that we shortly present in section 2 (for more information see [15]). Here, we emphasize the geometric context of the transformation, e.g. the Christoffel transformation depends on the Euclidean geometry of the ambient space while the Goursat transformation uses the interplay of the ambient space's Euclidean geometry and the conformal invariance of the notion of isothermic surfaces; the Darboux transformation is a transformation for surfaces in Möbius space while the T-transformation is only well defined for Möbius equivalence classes of isothermic surfaces. Then, we discuss the interrelations of these transformations in terms of several "permutability theorems" — some of the proofs become very compact in the quaternionic calculus.

In section 4, we interpret certain results on cmc-1 surfaces in hyperbolic space in terms of transformations of isothermic surfaces: the Umehara-Yamada deformation of minimal into cmc-1 surfaces is identified as the T-transformation and a unified version of the Weierstrass type representations for both surface classes is given in terms of quaternions. In this context, we give a Möbius geometric characterization for cmc-1 and minimal surfaces — in fact, the Umehara-Yamada perturbation families of cmc-1 surfaces appear as spectral families of isothermic surfaces. In terms of Möbius geometry, this means that the induced metric of the central sphere congruence has constant Gauss curvature 1. Finally, we introduce a second Weierstrass type representation for cmc-1 surfaces in H^3 in terms of the Darboux transformation, and discuss its relation with the notion of duality for cmc-1 surfaces.

As all the transformations under discussion require an integration they generally do not preserve periods, and may develop singularities, too. In this respect, our results have to be understood as local results. However, we formulate most of them in an invariant way in order to make the theory we present more accessible for further research on the global geometry of isothermic surfaces and their transformations. For example, it should be an interesting task to examine to what extent the Darboux and T-transformations respect extrinsic symmetries of surfaces, or how they behave in the neighbourhood of an isolated umbilic.

As mentioned above, many of the results of this paper can be formulated in a very elegant way using

2. The Quaternionic formalism

for Möbius differential geometry, as introduced in [15] (cf.[22],[25]) — some results even *depend* on that approach. On the other hand, the classical approach of considering Möbius geometry as a subgeometry of *real* projective geometry is the better known formalism and allows a more direct treatment of certain aspects in local differential geometry. In this section, we intend to discuss the relation between the two approaches briefly, focussing on 3-dimensional geometry[1].

Classically, the conformal n-sphere S^n is embedded into the real projective $(n+1)$-space $I\!\!RP^{n+1}$. Then, the Möbius transformations of S^n become the projective transformations of $I\!\!RP^{n+1}$ that fix $S^n \subset I\!\!RP^{n+1}$ as an absolute quadric. Up to scaling, there is a unique Lorentz scalar product on the space $I\!\!R_1^{n+2}$ of homogeneous coordinates such that the points of S^n are exactly the isotropic lines of that Lorentz product. Via orthogonality (polarity), hyperspheres can be identified with spacelike lines in $I\!\!R_1^{n+2}$, and the Möbius transformations appear as pseudo orthogonal transformations (cf.[3]).

On the other hand, the orientation preserving Möbius transformations of the conformal 4-sphere $I\!\!H \cup \{\infty\}$ can be identified with the projective transformations of the quaternionic projective line $I\!\!HP^1$ — just as the (orientation preserving) Möbius transformations of the conformal 2-sphere $C\!\!\!\!/ \cup \{\infty\}$ can be identified with the projective transformations of $C\!\!\!\!/ P^1$. Thus, in homogeneous coordinates[2] $v \in I\!\!H^2$ for points $vI\!\!H \in I\!\!HP^1 \cong S^4$, the orientation preserving Möbius transformations appear as linear transformations $v \mapsto Av$ with $A \in Gl(2, I\!\!H)$. Or, in affine coordinates $x \simeq (x, 1)^t$, the Möbius transformations become fractional linear transformations $x \mapsto (ax + b)(cx + d)^{-1}$ on $I\!\!H \cong I\!\!R^4$.

The relation between the two models can be established using the space of *quaternionic hermitian forms* $s : I\!\!H^2 \times I\!\!H^2 \to I\!\!H$ on $I\!\!H^2$,

$$s(u, v\lambda + w\mu) = s(u, v)\lambda + s(u, w)\mu \quad \text{and} \quad s(w, v) = \overline{s(v, w)}.$$

Fixing a basis (e_1, e_2) of the quaternionic plane $I\!\!H^2$, any quaternionic hermitian form is determined by its values $s(e_1, e_1), s(e_2, e_2) \in I\!\!R$ and $s(e_1, e_2) \in I\!\!H$. Thus, the space of quaternionic hermitian forms is a 6-dimensional real vector space. Endowed with the determinant as a quadratic form[3],

$$\langle s, s \rangle := -\det \begin{pmatrix} s(e_1, e_1) & s(e_1, e_2) \\ s(e_2, e_1) & s(e_2, e_2) \end{pmatrix} = |s(e_1, e_2)|^2 - s(e_1, e_1) \cdot s(e_2, e_2),$$

this space becomes the 6-dimensional Minkowski space $I\!\!R_1^6$ of the classical model for 4-dimensional Möbius geometry: space- and lightlike lines of quaternionic

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[1] For a more comprehensive treatment — particularly for the 4-dimensional setting that also appears natural when using quaternions — the reader is referred to [15].

[2] We consider $I\!\!H^2$ as a *right* vector space over the quaternions.

[3] Indeed, the Lorentz product introduced this way is only well defined up to a scaling: a change of basis $(e_1, e_2) \to (\tilde{e}_1, \tilde{e}_2)$ results in a rescaling of $\langle s, s \rangle \to \lambda^2 \langle s, s \rangle$ (cf.[15]). This is the same scaling ambiguity that arises in the classical construction, above.

hermitian forms can be identified with their null cones in $I\!\!H^2$ to encode hyperspheres $s \subset I\!\!H P^1$ and points $p \in I\!\!H P^1$ (cf.[15],[22]). Moreover, using the Study determinant $\mathcal{D}$ (cf.[1]) for quaternionic 2×2-matrices, the special linear group

$$Sl(2, I\!\!H) := \{M \in M(2 \times 2, I\!\!H) \,|\, \mathcal{D}M = 1\}$$

acts by isometries on $I\!\!R_1^6$ via $(M, s) \to M \cdot s := s(M^{-1}., M^{-1}.)$. In fact, $Sl(2, I\!\!H)$ is the universal cover of the identity component of the Möbius group.

As hyperspheres $s \subset I\!\!H P^1$ are encoded as spacelike lines of quaternionic hermitian forms, the Möbius group of the conformal 3-sphere can be obtained as the subgroup of $Sl(2, I\!\!H)$ fixing a given quaternionic hermitian form, say[4]

$$S^3 \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In affine coordinates, this choice corresponds to the imaginary quaternions as Euclidean 3-space, $\mathrm{Im}I\!\!H \cong I\!\!R^3$: the null cone of the above form is

$$S^3 = \{\begin{pmatrix} h \\ 1 \end{pmatrix} \cdot I\!\!H \,|\, h \in \mathrm{Im}I\!\!H\} \cup \{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot I\!\!H\} \subset I\!\!H P^1.$$

In this setting, the Lie algebra of the Möbius group of S^3 becomes

$$\mathfrak{m} = \{\begin{pmatrix} r + a & b \\ c & -r + a \end{pmatrix} \,|\, r \in I\!\!R; a, b, c \in \mathrm{Im}I\!\!H\} \subset \mathfrak{sl}(2, I\!\!H).$$

Space- and lightlike quaternionic hermitian forms that are perpendicular to the above fixed form representing S^3 can now be identified with 2-spheres and points, respectively, in the conformal 3-sphere S^3.

Given a surface $f : M^2 \to S^3$, a congruence of 2-spheres $s : M^2 \to I\!\!R_1^6$ in S^3 is said to be *enveloped* by f if each point $f(p)$ lies on the corresponding sphere $s(p)$, and if the tangent planes $d_p f(T_p M) = T_{f(p)} s(p)$ coincide. As discussed, incidence is encoded by $s(p)(f(p), f(p)) = 0$ in terms of homogeneous coordinates[5] $f : M^2 \to I\!\!H^2$. A simple calculation (cf.[15]) shows that

Lemma. *An immersion $f : M^2 \to S^3$ envelopes a congruence $s : M^2 \to I\!\!R_1^6$ of 2-spheres in S^3 if and only if*

$$s(f, f) \equiv 0 \quad \text{and} \quad s(df, f) + s(f, df) \equiv 0. \tag{1}$$

Fixing a basis (e_1, e_2) of $I\!\!H^2$, a map $F : M^2 \to \mathrm{Sl}(2, I\!\!H)$ will be called an *adapted frame* of an immersion $f : M^2 \to S^3$ if $f = Fe_1$. And, F is an adapted frame for a *surface pair* $f, \hat{f} : M^2 \to S^3$ if $f = Fe_1$ and $\hat{f} = Fe_2$. Note, that for a

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[4] Fixing one of the two length 1 representatives of the spacelike line amounts to fixing an orientation on the conformal 3-sphere S^3.

[5] We will use the same letter f for the immersion into $S^3 \subset I\!\!H P^1$ and for its homogeneous coordinates into $I\!\!H^2$.

surface pair $(f, \hat{f}) \simeq F$ the fact that F is invertible is equivalent to $f(p) \neq \hat{f}(p)$ at all points $p \in M^2$. Denoting $s_f, s_{\hat{f}} \in {I\!\!R}^6_1$ two generators of the lightlike lines of quaternionic hermitian forms corresponding to f and $\hat{f}$, respectively, $s_f(f, f) = 0$ together with $|s_f|^2 = 0$ implies that also $s_f(f, \hat{f}) = 0$. Thus, choosing a suitable scaling for s_f and $s_{\hat{f}}$, we have

$$F^{-1} \cdot s_f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F^{-1} \cdot s_{\hat{f}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2}$$

If $s : M^2 \to {I\!\!R}^6_1$ is a sphere congruence in S^3 containing the points of both surfaces of a surface pair $(f, \hat{f})$, then the first condition in (1) yields

$$F^{-1} \cdot s = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$$

with a function $c : M^2 \to \text{Im}{I\!\!H}$ into the imaginary quaternions. Given three such sphere congruences, say $s_i, s_j, s_k : M^2 \to {I\!\!R}^6_1$ defined by

$$F^{-1} \cdot s_i = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad F^{-1} \cdot s_j = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \quad F^{-1} \cdot s_k = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}, \tag{3}$$

a classical frame for Möbius geometry in S^3 (cf.[3]) is obtained: it consists of three congruences of orthogonally intersecting spheres s_i, s_j, and s_k, and the maps $s_f \cong f$ and $s_{\hat{f}} \cong \hat{f}$ describing their intersection points.

Since $d(F^{-1} \cdot s_c) = 0$ for all the above defined frame vectors s_c, one calculates $ds_c = -s_c(., \Phi.) - s_c(\Phi., .)$ with the connection form $\Phi = F^{-1}dF$. Using this equation allows to write the structure equations of the classical frame in terms of the components of Φ — thus obtaining a "translation table" between the classical and the quaternionic setting (in the codimension 1 case):

$$\Phi = \begin{pmatrix} \langle f, d\hat{f} \rangle + \frac{\langle ds_j, s_k \rangle}{2} i + \frac{\langle ds_k, s_i \rangle}{2} j + \frac{\langle ds_i, s_j \rangle}{2} k & \langle ds_i, \hat{f} \rangle i + \langle ds_j, \hat{f} \rangle j + \langle ds_k, \hat{f} \rangle k \\ \langle df, s_i \rangle i + \langle df, s_j \rangle j + \langle df, s_k \rangle k & \langle df, \hat{f} \rangle + \frac{\langle ds_j, s_k \rangle}{2} i + \frac{\langle ds_k, s_i \rangle}{2} j + \frac{\langle ds_i, s_j \rangle}{2} k \end{pmatrix}.$$

For example, the surface $f = Fe_1$, or $\hat{f} = Fe_2$, will envelope the sphere congruence s_i if and only if the i-component $\langle df, s_i \rangle = 0$, or $\langle d\hat{f}, s_i \rangle = 0$ respectively, in the corresponding off-diagonal form of Φ vanishes (cf.(1)).

At this point, we are prepared to collect (and reformulate) some results on

3. Transformations of Isothermic surfaces

Classically, an immersed surface $f : M^2 \to {I\!\!R}^3 \cong \text{Im}{I\!\!H}$ is called "isothermic" if it carries conformal curvature line coordinates $(x, y) : M^2 \to {I\!\!R}^2$, i.e.

$$\begin{aligned} \langle df, df \rangle &= e^{2u}(dx^2 + dy^2), \\ \langle df, dn \rangle &= -e^{2u}(k_1 dx^2 + k_2 dy^2), \end{aligned}$$

where $n : M^2 \to S^2$ is the normal field of f and $u : M^2 \to \mathbb{R}$ is a suitable real valued function. This definition obviously runs into problems when umbilics are present — and, since we are planning to consider sphere pieces as isothermic surfaces, we look for a more appropriate definition. On the tangent bundle of an immersed surface $f : M^2 \to \mathrm{Im}\mathbb{H}$, (quaternionic) left multiplication with the unit normal field $n : M^2 \to S^2$ yields a 90^0-rotation; the pull back of this structure provides a complex structure J on M^2: $df(Jx) = n \cdot df(x)$. Clearly, the immersion f is conformal with respect to this complex structure. It is well known that, away from umbilics, the existence of conformal curvature line coordinates is equivalent to the fact that the Hopf differential η of a conformal immersion f is a real multiple of a holomorphic quadratic differential q on M^2: $\eta = \varrho q$ with $\varrho : M^2 \to \mathbb{R}$. Thus, we give the definition of an isothermic immersion of a Riemann surface with "prescribed" principal curvature lines:

Definition (isothermic immersion). *A Riemann surface M^2 equipped with a holomorphic quadratic differential[6) q is called a polarized surface; a conformal immersion $f : (M^2, q) \to \mathrm{Im}\mathbb{H}$ of a polarized surface is called isothermic if its Hopf differential $\eta = \varrho q$ is a real multiple, $\varrho : M^2 \to \mathbb{R}$, of q.*

3.1 The Christoffel transformation

If two non homothetic immersions $f, f^* : M^2 \to \mathrm{Im}\mathbb{H}$ induce the same conformal structure and have parallel normal fields then [10] they are either associated minimal surfaces ($n^* = n$), or $n^* = -n$ and both surfaces are isothermic. The latter case can be characterized (cf.[15]) by the equation

$$df \wedge df^* = 0. \tag{4}$$

On the other hand, given an isothermic surface in terms of isothermic curvature line coordinates, $f : (\mathbb{C}, dz^2) \to \mathrm{Im}\mathbb{H}$, $df^* := f_x^{-1}dx - f_y^{-1}dy$ defines, at least locally, a second surface satisfying $df \wedge df^* = 0$. Obviously, the surface f^* is only determined up to homothety (and translation) by (4). Canonically trivializing the complex line bundle span$\{1, n\} \cong M^2 \times \mathbb{C}$ and using the above scaling of f^*, one obtains the polarization q back:

$$df \cdot df^* = (dx^2 - dy^2) + 2n \, dxdy \simeq dz^2.$$

In fact, if $\tau : TM^2 \to \mathrm{Im}\mathbb{H}$ is a 1-form with values in the imaginary quaternions then, $df \wedge \tau = 0$ if and only if $\tau(Jx) + n\tau(x) = \tau(Jx) - \tau(x)n = 0$ if and only if $(df \cdot \tau)((a + bJ)x) = (a + bn)^2(df \cdot \tau)(x)$ and $n\tau(x) + \tau(x)n = 0$ for any $x \in TM$:

Lemma. *Let $f : M^2 \to \mathrm{Im}\mathbb{H}$ be a conformal immersion of a Riemann surface and $\tau : TM \to \mathrm{Im}\mathbb{H}$ a 1-form with values in the imaginary quaternions. Then $df \cdot \tau : TM \to \mathrm{span}\{1, n\}$ defines a quadratic differential if and only if $df \wedge \tau = 0$.*

[6) In order to establish a theory of "globally isothermic surfaces" it seems too restrictive to ask q to be holomorphic — for example, with this assumption the ellipsoid is no longer isothermic [18]. In order to give a satisfactory global definition, the consequences of various possible assumptions on q have still to be worked out.

This lemma allows us (cf.[18]) to define *the* Christoffel transform of a polarized isothermic surface (up to translation), i.e. to fix the scaling of f^*:

Definition (Christoffel transformation). *Let $f : (M^2, q) \to \text{Im}\mathbb{H}$ be an isothermic immersion of a polarized Riemann surface. Then $Cf : M^2 \to \text{Im}\mathbb{H}$ is called Christoffel transform of f if $df \cdot dCf \simeq q$.*

The Christoffel transform Cf of an isothermic immersion f is unique up to translations and it is isothermic with respect to the polarization $\bar{q} = dCf \cdot df$. The Christoffel transform of Cf is the original surface, $f = C^2 f$.

As an example, we consider a minimal surface $f : M^2 \to \text{Im}\mathbb{H}$ in Euclidean 3-space: since $df \wedge dn = -H df \wedge df$, minimal surfaces are exactly those isothermic surfaces whose Christoffel transforms are totally umbilic (cf.[16]). As all the surfaces f_t in the associated family, $df_t = (\cos t + \sin t \cdot n) df$, have the same Gauss map the additional information provided by the (holomorphic) Hopf differential $df \cdot dn$ as polarization is required to reconstruct the original surface from its Christoffel transform (Gauss map).

3.2 The Goursat transformation

is classically defined for minimal surfaces, only: the action of the complex orthogonal group $O(3, \mathbb{C})$ on the holomorphic null curve describing an (associated family of) minimal surfaces provides a 3-parameter family of nontrivial transformations. For the Gauss map (Christoffel transform) of the minimal surface, these correspond to nontrivial Möbius transformations. Obviously, this type of transformation can be generalized to isothermic surfaces — making use of the fact that the notion of isothermic surface is conformally invariant while the Christoffel transformation depends on Euclidean geometry of the ambient space:

Definition (Goursat transformation). *Let $f : (M^2, q) \to \text{Im}\mathbb{H}$ be isothermic, and let $M : \text{Im}\mathbb{H} \cup \{\infty\} \to \text{Im}\mathbb{H} \cup \{\infty\}$ be a Möbius transformation. Then an immersion $Gf := CMCf$, i.e. the Christoffel transform of the Möbius transformed Christoffel transform of f, is called a Goursat transform of f.*

The Goursat transformations form a group acting on isothermic immersions of a polarized Riemann surface since $C^2 = id$. If the Möbius transformation M is a similarity, then the corresponding Goursat transform Gf of an isothermic immersion is clearly similar to the original immersion f. If the Möbius transformation M is "essential", however, Gf will generally be not Möbius equivalent to the original surface. In this way, the Goursat transforms of an isothermic immersion f provide a 3-parameter family of non Möbius equivalent isothermic immersions. Thus, we may restrict the attention to essential Möbius transformations — that are (up to similarity) of the form $x \to (x - m)^{-1}$ with $m \in \text{Im}\mathbb{H}$:

Lemma. *Let $x \to M(x) = (x - m)^{-1}$, $m \in \text{Im}\mathbb{H}$, be an essential Möbius transformation, and $f : (M^2, q) \to \text{Im}\mathbb{H}$ an isothermic immersion. Then [15],*

$$dGf = -(Cf - m) \cdot df \cdot (Cf - m). \tag{5}$$

Consequently, the Goursat transformation appears as a special type of spin transformation [21]. As an example, we obtain the classical Weierstrass representation of minimal surfaces: given a meromorphic function $g : M^2 \to \mathbb{C}$ and a holomorphic differential $\omega : TM^2 \to \mathbb{C}$, the two maps[7] $\int \omega j, -jg : M^2 \to \mathbb{C}j$ form a (trivial) Christoffel pair with respect to the polarization $q = \omega \cdot dg$. With the stereographic projection $\mathbb{C}j \ni x \to -i - 2(i + x)^{-1} \in S^2$, (5) yields the representation

$$df = \frac{1}{2}(i - jg)\omega j(i - jg) \tag{6}$$

of a minimal surface $f : M^2 \to \mathrm{Im}\mathbb{H}$ with Weierstrass data (g, ω) — cf.[15].

3.3 The Darboux transformation

A sphere congruence enveloped by two surfaces induces a point-to-point correspondence between the two surfaces. If, under this correspondence, the induced
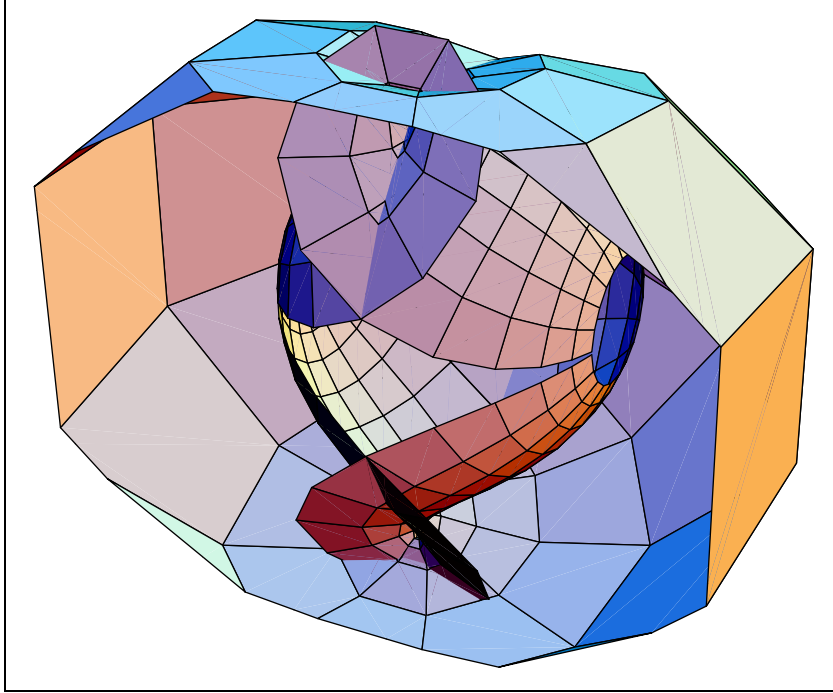

Fig. 1. A D-transform of a spherical net: cmc-1 in H^3

metrics on both surfaces are conformally equivalent and their curvature lines do correspond, then [12] either the two surfaces are Möbius equivalent or, both surfaces are isothermic. In terms of parametrizations $f, \hat{f} : M^2 \to \mathrm{Im}\mathbb{H}$, the

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[7] Here, we do not worry about periodicity of $\int \omega$, we *assume* the integral to exist.

latter situation can be characterized by the equation

$$df \wedge (f - \hat{f})^{-1} d\hat{f} = 0. \tag{7}$$

Comparison with (4) leads (cf.[16]) to the following

Lemma. *Let $f : M^2 \to \mathrm{Im}\,I\!H$ be an immersion. If there is a second immersion $\hat{f} : M^2 \to \mathrm{Im}\,I\!H$ satisfying (7) then f is isothermic with $q = df \cdot dCf$ where*

$$\lambda dCf = (\hat{f} - f)^{-1} d\hat{f}(\hat{f} - f)^{-1} \tag{8}$$

with some $\lambda \in I\!R \setminus \{0\}$ defines the Christoffel transform of f.

Rewriting (8) yields the Riccati equation $d(\hat{f} - f) = (\hat{f} - f)\lambda dCf(\hat{f} - f) - df$ for the difference vector field of f and $\hat{f}$ (cf.[16]). Given an isothermic immersion $f : (M^2, q) \to \mathrm{Im}\,I\!H$, this Riccati equation is completely integrable, i.e. fixing the parameter λ and an initial condition there exists (at least locally) a solution $\hat{f}$. This allows us to define the Darboux transforms D_λ of an isothermic immersion:

Definition (Darboux transformation). *Let $f : (M^2, q) \to \mathrm{Im}\,I\!H$ be an isothermic immersion of a polarized Riemann surface. Then $D_\lambda f : M^2 \to \mathrm{Im}\,I\!H$ is called a λ-Darboux transform of f if*

$$d(D_\lambda f - f) = (D_\lambda f - f)\lambda dCf(D_\lambda f - f) - df. \tag{9}$$

The linear version $0 = dv + \Phi_\lambda v$, $v : M^2 \to I\!H^2$, with $\Phi_\lambda = \begin{pmatrix} 0 & \lambda dCf \\ df & 0 \end{pmatrix}$ of (9) is exactly Darboux's linear system [12]: given a solution $v = (v_1, v_2)^t$, the corresponding Darboux transform[8] of f is given by $D_\lambda f = f + v_2 v_1^{-1}$. In terms of homogeneous coordinates this reads $D_\lambda f = F_0 v$ with the canonical Euclidean frame $F_0 = \begin{pmatrix} f & 1 \\ 1 & 0 \end{pmatrix}$. This description reflects the invariance of the notion of a Darboux transform under the action of the Möbius group: if $M \in Sl(2, I\!H)$ is a Möbius transformation then $D_\lambda M f = M F_0 v = M D_\lambda f$.

3.4 The T-transformation

Obviously, the above 1-form $\Phi_\lambda : TM^2 \to \mathfrak{sl}(2, I\!H)$ satisfies the Maurer-Cartan equation $d\Phi_\lambda + \Phi_\lambda \wedge \Phi_\lambda = 0$. Consequently, it can (locally) be integrated to a frame $F_\lambda : M^2 \to Sl(2, I\!H)$, $dF_\lambda = F_\lambda \Phi_\lambda$.

Definition (T-transformation). *Let $f : (M^2, q) \to \mathrm{Im}\,I\!H$ be an isothermic immersion of a polarized Riemann surface, and let $F_\lambda : M^2 \to Sl(2, I\!H)$ be a frame with connection form $F_\lambda^{-1} dF_\lambda = \Phi_\lambda$,*

$$\Phi_\lambda = \begin{pmatrix} 0 & \lambda dCf \\ df & 0 \end{pmatrix}. \tag{10}$$

Then $T_\lambda f := F_\lambda e_1$ is called T_λ-transform, or spectral transform of f.

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[8] Choosing an initial condition $v(p_0) \in S^3$ the solution will stay in S^3, thus $D_\lambda f : M^2 \to \mathrm{Im}\,I\!H$.
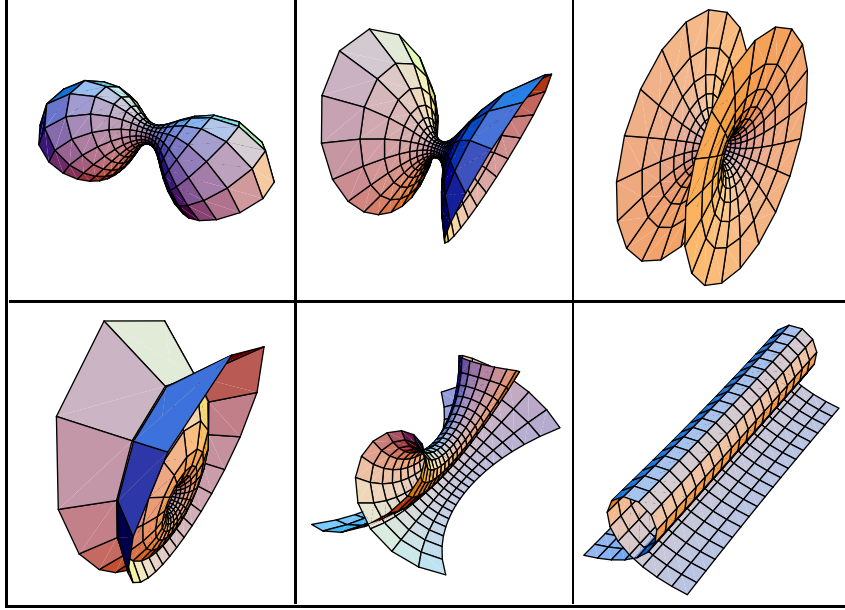
Fig. 2. T-transforms of the Catenoid: the Catenoid cousins

This way, the T_λ-transform $T_\lambda f$ of an isothermic immersion f is unique up to Möbius transformation. On the other hand, if $\tilde{f}$ is an essential Möbius transform of f, say $\tilde{f} = -(f-m)^{-1}$, then $C\tilde{f} = GCf$ is a Goursat transform of Cf and $d\tilde{F}_\lambda = \tilde{F}_\lambda \tilde{\Phi}_\lambda$ with $\tilde{F}_\lambda = F_\lambda \cdot \begin{pmatrix} (f-m)^{-1} & -1 \\ 0 & (f-m) \end{pmatrix}$. Consequently, $\tilde{F}_\lambda e_1$ and $F_\lambda e_1$ are just different homogeneous coordinates for the same surface in S^3: the T_λ-transformation is a well defined transformation for isothermic surfaces in Möbius geometry. In fact, the T_λ-transform of an isothermic immersion can be defined using *any* adapted frame: observing the effect of a Gauge transformation of adapted frames, $F_0 \to \tilde{F}_0 = F_0 A$ with $A e_1 = e_1 a$ and $a : M^2 \to I\!\!H$, on the corresponding connection forms shows that a suitable definition for $\tilde{\Phi}_\lambda$ provides a family of integrable connection forms corresponding to (10). Given an adapted frame $\tilde{F}_0 : M^2 \to Sl(2, I\!\!H)$ of an isothermic immersion $f : (M^2, q) \to \mathrm{Im}I\!\!H$,

$$\tilde{F}_0^{-1} d\tilde{F}_0 = \tilde{\Phi}_0 = \begin{pmatrix} \varphi & \hat{\psi} \\ \psi & \hat{\varphi} \end{pmatrix}, \quad \tilde{\Phi}_\lambda := \begin{pmatrix} \varphi & \hat{\psi} + \lambda \psi^* \\ \psi & \hat{\varphi} \end{pmatrix} \tag{11}$$

is integrable with $\tilde{F}_\lambda e_1 \simeq T_\lambda f$. Herein, ψ^* is defined — analogously to dCf — by the equation[9] $\psi \cdot \psi^* \simeq q$.

To understand the geometry of $T_\lambda f$, we first note that Φ_λ defined by (10) takes values in the Lie algebra of the Möbius group of $S^3 \subset I\!\!HP^1$. Consequently, $T_\lambda f$ can be assumed (after a suitable Möbius transformation) to take values in S^3, $T_\lambda f : M^2 \to S^3 = \mathrm{Im}I\!\!H \cup \{\infty\}$. Using affine coordinates, $F_\lambda = \begin{pmatrix} gb & \hat{g}\hat{b} \\ b & \hat{b} \end{pmatrix}$, yields

---

[9] Note that $\psi^\perp$ (similar to $df^\perp$) defines a complex line bundle over M^2 as $(\tilde{F}_0 e_1) I\!\!H : M^2 \to S^3$.

$dg = (\hat{g} - g)\hat{b}\,df\,b^{-1}$ and $d\hat{g} = (g - \hat{g})b\,\lambda dCf\,\hat{b}^{-1}$, hence $dg \wedge (g - \hat{g})^{-1}d\hat{g} = 0$. According to the above lemma, $g \simeq T_\lambda f$ is isothermic with Christoffel transform Cg given by $\mu dCg = (\hat{g} - g)^{-1}d\hat{g}(\hat{g} - g)^{-1}$. Calculating the corresponding quadratic differential, $\mu dg \cdot dCg = -\lambda[(g - \hat{g})\hat{b}]\,df \cdot dCf[(g - \hat{g})\hat{b}]^{-1} \simeq -\lambda q$, shows that the curvature lines of f and $T_\lambda f$ correspond[10]. Moreover, the Christoffel transform Cg of g can be scaled, $\mu = -\lambda$, such that $dg \cdot dCg \simeq q$. Then, $\hat{g} = D_{-\lambda}g$; on the other hand, $\hat{g} = T_\lambda Cf$ as the Gauge transformation $(e_1, e_2) \to (\frac{1}{\lambda}\sqrt{|\lambda|}e_2, \sqrt{|\lambda|}e_1)$ interchanges the roles of df and dCf in (10), i.e. $\hat{g} \simeq T_\lambda Cf$. This way, we obtain the first

3.5 Permutability

theorem (cf.[2]), $T_\lambda Cf = D_{-\lambda}T_\lambda f$. As the T_λ-transformation can be defined using *any* adapted frame, it is obvious that the 1-parameter family $\lambda \to T_\lambda f$ of T-transforms of an isothermic immersion f is a 1-parameter *group*: $T_\lambda T_\mu f = T_{\lambda+\mu}f$. Therefore, $T_\mu D_\lambda f = D_{\lambda-\mu}T_\mu f$.

Given an isothermic immersion $f : (M^2, q) \to \mathrm{Im}\,I\!H$, and a Darboux transform $D_\lambda f$ of f,

$$\begin{aligned}
\lambda dCf &= (D_\lambda f - f)^{-1}dD_\lambda f(D_\lambda f - f)^{-1}, \\
\lambda dCD_\lambda f &= (D_\lambda f - f)^{-1}df(D_\lambda f - f)^{-1}.
\end{aligned}$$

Hence, Cf and $CD_\lambda f$ can be positioned such that $\lambda(CD_\lambda f - Cf) = (D_\lambda f - f)^{-1}$. Then, $d(CD_\lambda f - Cf) = (CD_\lambda f - Cf)\lambda df(CD_\lambda f - Cf) - dCf$ showing (cf.[2],[16]) that $CD_\lambda f = D_\lambda Cf$. Combining this permutability theorem with the first, we obtain the following scheme:
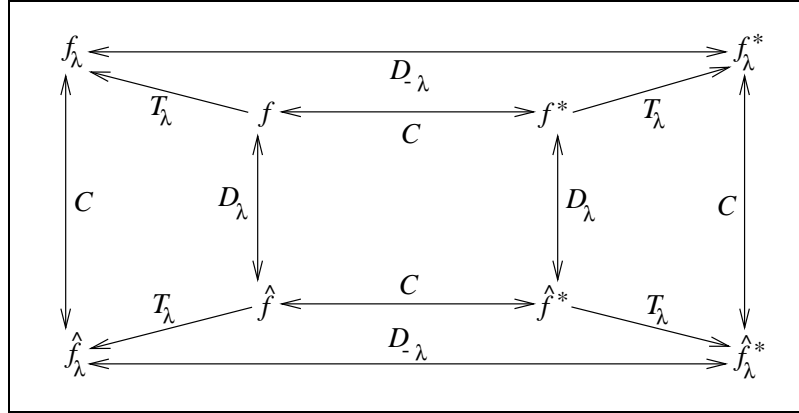

Fig. 3. A permutability theorem

Finally, consider two D_λ-transforms $(D_\lambda f)_1$ and $(D_\lambda f)_2$ of an isothermic immersion $f : (M^2, q) \to \mathrm{Im}\,I\!H$. The T_λ-transforms of these three surfaces are only well

---

[10] In fact, the family $\lambda \to T_\lambda f$ is the "conformal deformation" [19] of f (cf.[9]): isothermic surfaces are the only non rigid surfaces in Möbius geometry with prescribed curvature lines and conformal metric (Calapso potential), resp. with prescribed "conformal Hopf differential" [21].

defined up to Möbius transformation. But, after suitable Möbius transforma-
tions, $T_\lambda(D_\lambda f)_1$ is the Christoffel transform of $T_\lambda f$, as well as $T_\lambda(D_\lambda f)_2$ is — after
possibly other Möbius transformations. Thus, $T_\lambda(D_\lambda f)_2 = GT_\lambda(D_\lambda f)_1$: after a
T_λ-transformation, the "difference" between two D_λ-transforms of an isothermic
surface is given by a Goursat transformation.

4. Surfaces of constant mean curvature 1 in H^3

Thus far, we have not yet given any examples of Darboux or T-transforms. To
do so, it turns out to be convenient to use a suitably adapted frame for the two
envelopes $f, \hat{f} : M^2 \to S^3 \simeq s_1$ of a sphere congruence $s : M^2 \to I\!\!R_1^6$: given
that the curvature lines on both envelopes do correspond (s is "Ribaucour", $\hat{f}$
is a "Ribaucour transformation" of f), there are two congruences of 2-spheres
in S^3 such that each one intersects both surfaces orthogonally to one family of
curvature lines. Moreover, homogeneous coordinates $f, \hat{f} : M^2 \to I\!\!H^2$ can be
chosen such that these sphere congruences become s_j and s_k with respect to the
frame $F = (f, \hat{f}) : M^2 \to \mathrm{Sl}(2, I\!\!H)$, $s = s_i$ (cf.(3)), and that $\langle df, \hat{f} \rangle = 0$. This
way, the homogeneous coordinates of the two envelopes are fixed up to constant
rescaling. We will refer to such a frame as a "Ribaucour frame" for f and $\hat{f}$.

   If we additionally assume $f : (M^2, q) \to S^3$ to be isothermic, we can intro-
duce conformal curvature line parameters $z : M^2 \to \mathbb{C}$. Then, the connection
form of a Ribaucour frame, $\Phi = F^{-1}dF$, takes the form

$$\Phi = \begin{pmatrix} \frac{i}{2}[*du - (He^u dz + \hat{H}e^{-u}d\bar{z})j] & (\hat{\lambda}e^u dz - \lambda e^{-u}d\bar{z})j \\ e^u dz\, j & \frac{i}{2}[*du - (He^u dz + \hat{H}e^{-u}d\bar{z})j] \end{pmatrix} \quad (12)$$

with suitable real functions $u, H, \hat{H}, \lambda, \hat{\lambda} : M^2 \to I\!\!R$. The Maurer-Cartan equa-
tion for Φ, $0 = d\Phi + \Phi \wedge \Phi$, yields the Gauss equation

$$\Delta u = -H^2 e^{2u} + \hat{H}^2 e^{-2u} + 4e^{2u}\hat{\lambda}, \quad (13)$$

and the Codazzi equations

$$0 = \hat{\lambda}_{\bar{z}} e^u + \lambda_z e^{-u} \qquad \text{and} \qquad H_{\bar{z}} e^u = \hat{H}_z e^{-u}. \quad (14)$$

4.1 The D-transformation and cmc-1 surfaces in H^3

Now, f and $\hat{f}$ induce conformally equivalent metrics on M^2 if either $\lambda \equiv 0$ or
$\hat{\lambda} \equiv 0$ while the other is constant by (14). In the former case, the two surfaces, f
and $\hat{f}$, are Möbius equivalent; in the latter $\hat{f}$ is a Darboux transform of f (and
vice versa), and F is a "curved flat frame" (cf.[15]).

   The second surface, $\hat{f}$, of such a pair $F = (f, \hat{f})$ of Darboux transforms of
each other is totally umbilic if and only if $H \equiv 0$ in (12). Obviously, this is
equivalent to the fact that $\mathrm{tr}\langle df, ds \rangle = 0$ with respect to the induced conformal

structure $\langle df, df \rangle$ of f, characterizing the enveloped sphere congruence s as the "central sphere congruence" of f (cf.[3]). By the Codazzi equations, also $\hat{H}$ is constant; consequently, $\hat{H}\hat{f} + \lambda s =: s_\infty$ is constant, $ds_\infty = \hat{H}d\hat{f} + \lambda\, ds \equiv 0$, thus describing a constant 2-sphere (as long as $\lambda \neq 0$). As $\langle \hat{f}, s_\infty \rangle \equiv 0$, $\hat{f}$ takes values in that 2-sphere s_∞. If $\hat{H} \equiv 0$, also f takes values in s_∞ and a "Darboux pair of meromorphic functions" is obtained as a degenerate case. If $\hat{H} \neq 0$, we can assume $\hat{H} \equiv 1$ without loss of generality[11]. Interpreting the two components of

$$ S^3 \setminus s_\infty \cong \{ x \in I\!\!R_1^6 \,|\, \langle x, x \rangle = 0,\ \langle x, s_1 \rangle = 0,\ \langle x, s_\infty \rangle = -\tfrac{1}{2} \} $$

as hyperbolic spaces of sectional curvature $-4\lambda^2$ with infinity boundary s_∞, the sphere congruence s becomes a congruence of horospheres in one of these hyperbolic spaces, $H^3 \subset S^3$. Using the tangent plane congruence $t := s + 2\lambda f$ of f in hyperbolic space[12], the first and second fundamental form of f become

$$ \begin{array}{rcl} \langle df, df \rangle & = & e^{2u}\,(dx^2 + dy^2) \\ -\langle df, dt \rangle & = & -2\lambda e^{2u}\,(dx^2 + dy^2) \quad + \quad (dx^2 - dy^2) \end{array} \tag{15} $$

showing that f has constant mean curvature[13] $h \equiv -2\lambda$ (as a surface in hyperbolic space). Summarizing, we obtain the following

Theorem. *Let $\hat{f} : (M^2, \bar{q}) \to s_\infty \subset S^3$ be an isothermic immersion into a 2-sphere s_∞. Then, any Darboux transform $f : (M^2, q) \to S^3$ of $\hat{f}$ either takes values in s_∞, too, or parametrizes a surface of constant mean curvature h in a hyperbolic space H_k^3 of sectional curvature $k = -h^2$. In the latter case, the enveloped Ribaucour sphere congruence is the central sphere congruence of f, and $\hat{f}$ is the hyperbolic Gauss map of f.*

To confirm the last statement of the theorem, we just notice that the circles intersecting the central spheres s orthogonally in f and $\hat{f}$ are the hyperbolic geodesics orthogonal to f. Their intersection points with the infinity boundary s_∞ are given by $\hat{f}$ and $\tilde{f} := f + \frac{1}{2\lambda}s + \frac{1}{4\lambda^2}\hat{f}$. As f and $\hat{f}$ induce conformally equivalent metrics on M^2 (while f and $\tilde{f}$ do not) the claim follows (see [5]).

4.2 The T-transformation and cmc-1 surfaces in H^3

Having a closer look at the connection form

$$ \Phi_\lambda := \begin{pmatrix} \frac{i}{2}[*du - e^{-u}d\bar{z}\,j] & -\lambda e^{-u}d\bar{z}\,j \\ e^{u}dz\,j & \frac{i}{2}[*du - e^{-u}d\bar{z}\,j] \end{pmatrix} \tag{16} $$

---

[11] By a constant Gauge transformation $(f, \hat{f}) \to (\frac{1}{\hat{H}}f, \hat{H}\hat{f})$ and possibly interchanging the roles of the principal curvature directions, a new Ribaucour frame with (12) is obtained, where $u - \ln|\hat{H}| \to u$.

[12] As $\langle t, s_\infty \rangle \equiv 0$, the spheres t intersect the infinity boundary s_∞ orthogonally.

[13] Here, we could as well have argued that the mean curvature of f at any point coincides with the mean curvature $-2\langle s, s_\infty \rangle$ of the central sphere s at that point, measured in any space form.

of the above Ribaucour frame for the surface $f_\lambda := f : M^2 \to H^3_{4\lambda^2}$ of constant mean curvature -2λ and its hyperbolic Gauss map $\hat{f}_\lambda := \hat{f} : M^2 \to s_\infty =: s_\infty(\lambda)$ shows immediately[14] that $f_\lambda = T_\lambda f_0$ is a T_λ-transform of an isothermic surface $f_0 : (M^2, q) \to \mathrm{Im}I\!H$: sending $\lambda \to 0$, the infinity sphere $s_\infty(\lambda) \to \hat{f}_0 \equiv p_\infty$ degenerates to a point, $d\hat{f}_0 \equiv 0$, that we interprete as the point at infinity of Euclidean 3-space $\mathrm{Im}I\!H$. Since the central spheres s_0 of f_0 all contain the point p_∞ at infinity they are planes in $\mathrm{Im}I\!H$. Consequently, the surface f_0 is minimal in $\mathrm{Im}I\!H$. On the other hand, away from umbilics, every minimal immersion $f_0 : M^2 \to \mathrm{Im}I\!H$ carries conformal curvature line coordinates $z : M^2 \to \mathbb{C}$ (minimal immersions are isothermic) such that the Hopf differential $q = dz^2$. Thus, every minimal immersion has a frame with connection form Φ_0, and

Theorem. *The surfaces of constant mean curvature $h = -2\lambda$ in hyperbolic space of sectional curvature $k = -4\lambda^2$ are in one-to-one correspondence with the minimal surfaces in Euclidean 3-space via the T_λ-transformation.*

As all surfaces $f_\lambda = T_\lambda f_0 : M^2 \to H^3_{-4\lambda^2}$, $\lambda \in I\!R$, are isometric, $I_\lambda = I_0$, and their second fundamental forms satisfy $I\!I_\lambda = I\!I_0 - 2\lambda I_0$ the T_λ-transformation yields exactly the "Umehara-Yamada perturbation" [23] of minimal surfaces into constant mean curvature surfaces in hyperbolic space. Consequently, our previously discussed version (6) of the Weierstrass representation for minimal surfaces in Euclidean 3-space "perturbs" into a version of Bryant's Weierstrass type representation [5]: given Weierstrass data (g, ω) and $\lambda \in I\!R$ the corresponding surface f_λ of constant mean curvature -2λ in $H^3_{-4\lambda^2}$ (together with its hyperbolic Gauss map $\hat{f}_\lambda$, if $\lambda \neq 0$) or $I\!R^3$, respectively, is obtained by integrating the system

$$
\begin{array}{rcl}
df_\lambda & = & \hat{f}_\lambda \cdot [\frac{1}{2}(i - jg)\omega\, j(i - jg)], \\
d\hat{f}_\lambda & = & f_\lambda \cdot [-2\lambda(i - jg)^{-1} jdg(i - jg)^{-1}].
\end{array}
\tag{17}
$$

Remark. *The Gauss map $(i - jg)i(i - jg)^{-1}$ of the minimal surface f_0 is called the "secondary Gauss map" of f_λ, and f_0 its "minimal cousin" (cf.[5]).*

4.3 Isothermic surfaces of spherical type

The Gauss equation (13) for the connection form (16) associated with a surface $f_\lambda : M^2 \to H^3_{-4\lambda^2}$ of constant mean curvature -2λ (or, its minimal cousin $f_0 : M^2 \to I\!R^3$) reduces to the Liouville equation

$$
\Delta u = e^{-2u}.
\tag{18}
$$

Consequently, the induced metric $\langle ds, ds \rangle = e^{-2u}(dx^2 + dy^2)$ of the (central) sphere congruence s has constant Gauss curvature 1:

---

[14] Remember that the T-transformation is frame independent, see (11).

Definition (spherical type). *An isothermic immersion is called isothermic of spherical type if its central sphere congruence induces a metric of constant Gauss curvature 1.*

We will see that this is a (Möbius geometric) characterization for surfaces of constant mean curvature h in hyperbolic space of sectional curvature $k = -h^2$, and minimal surfaces in Euclidean space: it is well known [3] that a surface $f : M^2 \to S^3$ is isothermic if and only if its central sphere congruence s is Ribaucour. Therefore, we can associate a Ribaucour frame $F = (f, \hat{f})$ to f where $\hat{f}$ is the second envelope of the central sphere congruence. In the connection form (12), the fact that $s = s_i$ is the central sphere congruence of f is reflected by $H \equiv 0$. By (14), $\hat{H}$ is constant so that without loss of generality $\hat{H} \equiv 1$ (see footnote 11). Now, u satisfies the Liouville equation (18) if and only if $\hat{\lambda} \equiv 0$, if and only if the connection form (12) takes the form (16). Thus,

Theorem. *An isothermic immersion is isothermic of spherical type*
iff the second envelope of its central sphere congruence is a Darboux transform
    — this Darboux transform is then either totally umbilic, or a point —
iff the surface is either a surface of constant mean curvature h in a hyperbolic space of sectional curvature $k = -h^2$, or a minimal surface in Euclidean space.

Consequently, *every* surface $f : M^2 \to H_k^3$ of constant mean curvature h in a space of sectional curvature $k = -h^2$ can be obtained as a D_λ-transform, $\lambda = -\frac{h}{2}$, of a totally umbilic immersion $n_h : M^2 \to S^2 \cong \partial H_k^3$. This gives rise to another Weierstrass type representation for such surfaces: given Weierstrass data (g, ω), integration of Darboux's linear system[15]

$$
\begin{array}{rclcl}
dv_1 & + & \lambda \omega j \cdot v_2 & = & 0 \\
dv_2 & - & j dg \cdot v_1 & = & 0
\end{array}
\tag{19}
$$

yields a surface $f_\lambda^\# = -jg + v_2 v_1^{-1}$ of constant mean curvature $h = 2\lambda$ in $H_{-4\lambda^2}^3$ as a D_λ-transform of its hyperbolic Gauss map $n_h = -jg$ — as long as an initial condition for v is chosen such that $v_2 v_1^{-1} \notin \mathbb{C}j$. The secondary Gauss map of $f_\lambda^\#$ and its minimal cousin are obtained as T_λ-transforms of $-jg$ and $f_\lambda^\#$, respectively. Thus, comparing this second Weierstrass type representation for surfaces of constant mean curvature h in $H_{-h^2}^3$ with Bryant's, the roles of the hyperbolic and secondary Gauss maps of the surface are interchanged.

4.4 Duality for cmc-1 surfaces in H^3

In [24] M. Umehara and K. Yamada introduced a notion of "duality" for cmc-1 surfaces in hyperbolic space: given a cmc-1 surface $f : M^2 \to H^3$ with hyperbolic and secondary Gauss maps $n_h, n_s : M^2 \to S^2 \cong \partial H^3$, the "dual cmc-1

---

[15] Note that multiplying the second equation by j and substituting $jv_2 \to v_2$, this system can entirely be handled in the context of complex function theory: given a (complex) fundamental system *any* solution is obtained as a *quaternionic* superposition.

surface" $f^\# : M^2 \to H^3$ is obtained by interchanging the roles of the two Gauss maps, $n_h^\# = n_s$ and $n_s^\# = n_h$. However, the hyperbolic Gauss map n_h is only well defined up to Möbius transformation of S^2 since isometries of H^3 extend to Möbius transformations of $S^2 \cong \partial H^3$. Consequently, the dual cmc-1 surface $f^\#$ depends on the position of f in hyperbolic space; as there is a 3-parameter family of essential Möbius transformations of the 2-sphere, generally, there is a 3-parameter family of dual cmc-1 surfaces. In the context of transformations of isothermic surfaces, this duality relation occurs as a special case of the permutability theorem sketched in Fig. 3: as discussed above, the hyperbolic Gauss map $n_h = \hat{f}$ of a cmc-1 surface f is a $D_{\frac{1}{2}}$-transform of f where the enveloped sphere congruence is the central sphere congruence of f. The minimal cousin of f and its Gauss map (the secondary Gauss map of f) are obtained as $T_{\frac{1}{2}}$-transforms of f and $\hat{f}$, respectively. Now, the dual cmc-1 surface[16] $f^\# = \hat{f}_{\frac{1}{2}}^*$ is a $D_{-\frac{1}{2}}$-transform of $n_s = T_{\frac{1}{2}} n_h$, and $f^* = T_{\frac{1}{2}} f^\#$ is its minimal cousin. The 3-parameter ambiguity of the dual cmc-1 surface $f^\#$ is the 3-parameter ambiguity of the Darboux transformation, and the minimal cousins of different dual cmc-1 surfaces differ by a Goursat transformation.

We conclude our paper with a simple

4.5 Example:

consider $f(z) := -jz$ as an isothermic immersion of the polarized plane $(\mathbb{C}, d\bar{z}^2)$ into $\mathbb{C}j \subset \mathrm{Im}\,\mathbb{H}$. Its Christoffel transform is $Cf = zj : (\mathbb{C}, dz^2) \to \mathbb{C}j$. With

$$\Phi_\lambda = \begin{pmatrix} 0 & \lambda dCf \\ df & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -j \end{pmatrix} \begin{pmatrix} 0 & \lambda dz \\ dz & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix},$$

integration of the linear system $dF_\lambda = F_\lambda \Phi_\lambda$, $F_\lambda(0) = Id$, yields

$$F_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & -j \end{pmatrix} \begin{pmatrix} \cosh(\sqrt{\lambda}z) & \sqrt{\lambda}\sinh(\sqrt{\lambda}z) \\ \frac{1}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}z) & \cosh(\sqrt{\lambda}z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix}.$$

Thus, $T_\lambda f = -j\frac{1}{\sqrt{\lambda}}\tanh(\sqrt{\lambda}z)$ — or any Möbius transform of it. Its Christoffel transform $CT_\lambda f$ depends on the choice of representative for $T_\lambda f$: with the above choice, $CT_\lambda f = \frac{1}{2}[z + \frac{1}{2\sqrt{\lambda}}\sinh(2\sqrt{\lambda}z)]j$, any other choice yields a Goursat transform thereof. For example, stereographic projection of $T_\lambda f$ to the unit 2-sphere yields a family $\frac{1}{4}\{\mathrm{Re}[\frac{\cosh(2\sqrt{\lambda}z)-1}{\lambda}]i + [z + \frac{\sinh(2\sqrt{\lambda}z)}{2\sqrt{\lambda}}]j + j\frac{1}{\lambda}[z - \frac{\sinh(2\sqrt{\lambda}z)}{2\sqrt{\lambda}}]\}$ of minimal surfaces with the Catenoid at $\lambda = 1$ and the Enneper surface at $\lambda = 0$. The homogeneous coordinates of any Darboux transform $D_\lambda f = F_0 F_\lambda^{-1} v_0$ with constant $v_0 \in \mathbb{H}^2$. Choosing $D_\lambda f(0) \simeq v_0 \mathbb{H} \notin \mathbb{C}j \cup \{\infty\}$, $D_\lambda f$ is a surface of constant mean curvature 2λ in $H^3_{-4\lambda^2}$. For example, $v_0 = (1, -i)^t$ yields the surfaces $D_\lambda f(z) = -j\{z - [\frac{\sinh(\sqrt{\lambda}z)}{\sqrt{\lambda}} - \cosh(\sqrt{\lambda}z)k][\cosh(\sqrt{\lambda}z) - \sqrt{\lambda}\sinh(\sqrt{\lambda}z)k]^{-1}\}$ that have the above minimal surfaces as their minimal cousins.

---

[16] With respect to the original polarization, $f^\#$ has constant mean curvature -1; changing the direction of the normal field, $q \to -q$, yields mean curvature 1 (cf.[24]).

In order to determine the Darboux transforms of $T_\lambda f$ too, one has to integrate Darboux's linear system again with a connection form $\Phi_{\lambda,\mu} = \Phi_\mu(T_\lambda f)$. As the Darboux transformation is a Möbius geometric notion any choice of representative for $T_\lambda f$ will lead to the same (in terms of Möbius geometry) result. Thus, with the above choice $T_\lambda f = -j\frac{\tanh(\sqrt{\lambda}z)}{\sqrt{\lambda}}$, we obtain $F_{\lambda,\mu}e_1 = F_{\lambda+\mu}\frac{e_1}{\cosh(\sqrt{\lambda}z)}$ confirming that $T_\mu'T_\lambda f = T_{\lambda+\mu}f$. In the special case $\mu = -\lambda$ of our permutability theorem (see Figure 3), $F_{\lambda,0}F_{\lambda,-\lambda}^{-1} = [F_0 F_\lambda^{-1}]^{-1}$ so that the Darboux transforms $D_{-\lambda}T_\lambda f = -j\{\frac{\tanh(\sqrt{\lambda}z)}{\sqrt{\lambda}} - \frac{1}{\cosh(\sqrt{\lambda}z)}[z-k][\cosh(\sqrt{\lambda}z) - \sqrt{\lambda}\sinh(\sqrt{\lambda}z)(z-k)]^{-1}\}$ can be determined without further integration. Note that according to the permutability theorem all the surfaces $D_{-\lambda}T_\lambda f$ have the Enneper surface as their minimal cousin, and they are dual cmc surfaces of $D_\lambda f$ in the sense of [24].

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